Applications of the Löwenheim-Skolem theorem. Part I

Já dovedu všechno

K. P. Hart

Faculty EEMCS
TU Delft

Hejnice, 29. Leden, 2012: 10:30 – 11:20

Delft University of Technology

Outline

Delft University of Technology

The theorem

Just so we know what we are talking about.

Theorem

Let A be a structure for some language $\mathcal{L}$ and let X be a subset of A. There is an elementary substructure B of A of cardinality at most $\aleph_0 \cdot |X| \cdot |\mathcal{L}|$ such that $X \subseteq B$.

We need to explain a few terms: structure, language, elementary substructure,

Delft University of Technology

Language

We mean *mathematical* languages.

All languages have the usual logical symbols in their vocabulary:

$$\forall, \exists, \wedge, \vee, \rightarrow, \neg, =, (,)$$

Language

Each language has its own set of specific symbols:

$<$	'ordered sets'
$*, e$	'groups'
$\sqcap, \sqcup$	'lattices'
$+, \cdot, 1, 0$	'fields'
$\in$	'sets'

We'll use $\sqcap$ and $\sqcup$ for inf and sup as $\wedge$ and $\vee$ are already taken.
Often only the specific symbols are deemed part of the language.

Structure

A structure for a language, $\mathcal{L}$, consists of a set, X, together with an assignment that associates

- constants in $\mathcal{L}$ with elements of X
- function symbols in $\mathcal{L}$ with functions from some (finite) power of X to X
- predicate symbols in $\mathcal{L}$ with subsets of (finite) powers of X

Delft University of Technology

Examples

- A structure for $\{<\}$ consists of a set, X and one subset of X^2, associated with $<$.
- A structure for $\{*, e\}$ consists of a set, X and one function $f : X^2 \to X$ (for $*$) and one designated element x, associated with e.
- A structure for $\{\sqcap, \sqcup\}$ consists of a set, X, and two functions from X^2 to X (one for $\sqcap$, one for $\sqcup$).
- A structure for $\{+, \cdot, 1, 0\}$ consists of a set, X, two functions from X^2 to X (one for $+$, one for $\cdot$) and two designated elements, x and y, to go with 1 and 0.

TUDelft
Delft University of Technology

Theories

Some symbols come laden with history: consider this structure for the language $\{<\}$.

Underlying set: $\mathbb{Z}$

Interpretation for $<$: $\{\langle x, y\rangle : 7 \mid x - y\}$
i.e., $x < y$ means $x - y$ is divisible by 7.

This is unfortunate, but completely legal.

Theories

Usually we choose our symbols with some intent, e.g., $<$ should stand for 'less than'.
Therefore, in this case we commonly stipulate that we only consider structures where the interpretation of $<$ satisfies a few properties/formulas:

- $(\forall x)\bigl(\neg(x < x)\bigr)$
- $(\forall x)(\forall y)\bigl((x < y) \vee (y < x) \vee (x = y)\bigr)$
- $(\forall x)(\forall y)(\forall z)\Bigl(\bigl((x < y) \wedge (y < z)\bigr) \to (x < z)\Bigr)$

These formulas together form the *theory* of linear orders.
That's what a *mathematical* theory is: a set of formulas in a certain language.

The language $\{\in\}$

A structure for $\{\in\}$ consists of a set, X and one subset of X^2, associated with $\in$.

Wait, wait! What?

Yes, $\{\in\}$ specifies a language, just like the others and we don't treat it differently.

A structure, X, with interpretation E for the symbol $\in$ satisfies the Axiom of Extensionality if

$$(\forall x \in X)(\forall y \in X)(\forall z \in X)\Big(\big((z\,E\,x) \leftrightarrow (z\,E\,y)\big) \rightarrow (x = y)\Big)$$

The language $\{\in\}$

Very often, the interpretation of $\in$ will be $\in$ itself, i.e.,

$$E = \{\langle x, y\rangle : x \in y\}$$

And you know that Extensionality holds if

$$(\forall x \in X)(\forall y \in X)\big((x \cap X = y \cap X) \rightarrow (x = y)\big)$$

This can be confusing at first but if you master this you'll have a very powerful tool in your hands.

See also this afternoon.

Models

A *model* for a theory in a certain language is a structure for that language in which all formulas from the theory are true.

Thus, a model for the theory of linear orders is, surprise, a linearly ordered set.

Delft University of Technology

Models

A model for Set Theory would be a set with a specified relation in which all the axioms of ZFC are true.

Gödel's incompleteness theorem says we can't prove their existence in ZFC itself but that is, for applications, not important.

There are nice sets that, with $\in$ interpreted by itself, satisfy a lot of Set Theory and applications of the Löwenheim-Skolem theorem to these structures yield many interesting results.

Delft University of Technology

Substructure

This is easy, you can think of a definition yourself.

Think of *sub*order, *sub*group, *sub*field, *sub*lattice, ...

Elementary substructure

This is quite different; an elementary substructure is much, much richer than just a substructure.

Definition

Let X be a structure for a language $\mathcal{L}$ and $Y \subseteq X$ a substructure. We say that Y is an *elementary* substructure of X if for every formula φ of $\mathcal{L}$ of the form $(\exists y)\psi(y, x_1, \ldots, x_n)$ and all $a_1, \ldots, a_n \in Y$: if $\varphi(a_1, \ldots, a_n)$ is true in X then there is an $a \in Y$ such that $\psi(a, a_1, \ldots, a_n)$ is true in X.

You can then prove that $\psi(a, a_1, \ldots, a_n)$ is, in fact, true in Y.

Delft University of Technology

Examples

Notation: $Y \prec X$ means Y is an elementary substructure of X.
The field $\mathbb{Q}$ is not an elementary subfield of $\mathbb{R}$.
$(\exists x)(x \cdot x = 1 + 1)$ is true in $\mathbb{R}$, but there is no $q \in \mathbb{Q}$ such that $q \cdot q = 1 + 1$ is true in $\mathbb{R}$ (nor in $\mathbb{Q}$).

Delft University of Technology

Examples

The linearly ordered set $\mathbb{Q}$ is an elementary substructure of the linearly ordered set $\mathbb{R}$.

This seems obvious, both look superficially the same, but the proof is far from trivial.
It requires a thorough investigation of the structure of the formulas of the language and the kinds of subsets of $\mathbb{R}$ (or $\mathbb{Q}$) that they decribe.

Delft University of Technology

And yet . . .

The Löwenheim-Skolem theorem says that there are very many elementary substructures of a given structure.

The difference is that we have no idea how to describe them explicitly but, for us, that is not important.

The proof of the Löwenheim-Skolem theorem involves the Axiom of Choice (the simultaneous choice of the as, for all φ and $a_1, \ldots, a_n$).

Delft University of Technology

Skolem functions

The choice functions mentioned above are known as Skolem functions.
Every formula φ of the form $(\exists y)\psi(y, x_1, \ldots, x_n)$ determines a function $F_\varphi : X^n \to \mathcal{P}(X)$, defined by

$$F_\varphi(x_1, \ldots, x_n) = \{y : \psi(y, x_1, \ldots, x_n)\}$$

A Skolem function for φ is a choice function for F_φ (fix some $x_0 \in X$ in advance and set $f_\varphi(x_1, \ldots, x_n) = x_0$ whenever $F_\varphi(x_1, \ldots, x_n) = \emptyset$).

The proof

Once the Skolem functions are in place we can take, given $Y \subseteq X$, the smallest subset Z of X that contains Y and is closed under all Skolem functions.

A Factorization Theorem

Theorem (Mardešić)

Let $f : X \to Y$ be a continuous surjection between compact Hausdorff spaces. Then there are a compact Hausdorff space Z and continuous maps $h : Z \to Y$ and $g : X \to Z$ such that $f = h \circ g$, $\dim Z = \dim X$ and $w(Z) = w(Y)$.

How does this follow from the Löwenheim-Skolem theorem?

Delft University of Technology

Apply the theorem

The family of, 2^X, of all closed subsets of X is a lattice, with $\cap$ and $\cup$ as its binary operations.

The family $\mathcal{Y} = \{f^{-1}[F] : F \in 2^Y\}$ is a sublattice of 2^X.

Since $\bigl|\{\cap, \cup\}\bigr| = 2$ the Löwenheim-Skolem theorem gives us an elementary sublattice $\mathcal{Z}$ of 2^X such that $\mathcal{Y} \subseteq \mathcal{Z}$ and $|\mathcal{Z}| = |\mathcal{Y}|$.

Make a space out of $\mathcal{Z}$

From $\mathcal{Z}$ we create our space Z:

- Z is the set of ultrafilters on $\mathcal{Z}$
- $\{F^* : F \in \mathcal{Z}\}$ is a base for the *closed* sets of Z, where $F^* = \{z \in Z : F \in z\}$.

Delft University of Technology

Z is compact Hausdorff

That Z is compact follows almost by construction.

That Z is Hausdorff follows by elementarity.
Let $z_1 \neq z_2$; use 'ultra' to find $F_1 \in z_1$ and $F_2 \in z_2$ such that $F_1 \cap F_2 = \emptyset$.
In 2^X the following is true:

$$(\exists G)(\exists H)\bigl((G \cup H = X) \wedge (G \cap F_1 = \emptyset) \wedge (H \cap F_2 = \emptyset)\bigr)$$

Why? Because X is normal.

Z is compact Hausdorff

Note $X, \emptyset \in \mathcal{Z}$, by elementarity.
(Use $(\exists! x)(\forall y)(y \cup x = x)$ and $(\exists! x)(\forall y)(y \cap x = x)$)

So we can find $G, H \in \mathcal{Z}$ that make the same formula true.

Now note that $Z \setminus G^*$ and $Z \setminus H^*$ are disjoint neighbourhoods of z_1 and z_2 respectively.

Delft University of Technology

Dimension

Definition (Lebesgue)

$\dim X \leqslant n$ if every finite open cover has a (finite) open refinement of order at most $n+1$
(i.e., every $n+2$-element subfamily has an empty intersection).

There is a convenient characterization.

Theorem (Hemmingsen)

$\dim X \leqslant n$ iff every $n+2$-element open cover has a shrinking with an empty intersection.

Delft University of Technology

Dimension

Here is Hemmingsen's characterization of $\dim X \leqslant n$ reformulated in terms of closed sets and cast as a formula, δ_n, in the language of lattices

$$
\begin{aligned}
&(\forall x_1)(\forall x_2)\cdots(\forall x_{n+2})(\exists y_1)(\exists y_2)\cdots(\exists y_{n+2}) \\
&\qquad\qquad\Big[(x_1 \sqcap x_2 \sqcap \cdots \sqcap x_{n+2} = 0) \rightarrow \\
&\qquad\Big((x_1 \leqslant y_1) \wedge (x_2 \leqslant y_2) \wedge \cdots \wedge (x_{n+2} \leqslant y_{n+2}) \\
&\qquad\qquad \wedge (y_1 \sqcap y_2 \sqcap \cdots \sqcap y_{n+2} = 0) \\
&\qquad\qquad\qquad \wedge (y_1 \sqcup y_2 \sqcup \cdots \sqcup y_{n+2} = 1)\Big)\Big].
\end{aligned}
$$

Dimension

Elementarity: 2^X and $\mathcal{Z}$ satisfy δ_n for exactly the same values of n. For, $\neg\delta_n$ also determines a Skolem function (a constant one).

Now apply some topology (swelling and shrinking) to conclude that the full family, 2^Z, of closed sets of Z satisfies δ_n for exactly the same values of n as $\mathcal{Z}$ does.

We conclude $\dim X = \dim Z$.

Weight

What about the weight of Z?

Probably larger than that of Y, but we could have started with a base, $\mathcal{C}$, for the closed sets of Y, of cardinality $w(Y)$, that is also a lattice, and apply the Löwenheim-Skolem theorem to $\{f^{-1}[F] : F \in \mathcal{C}\}$.

This yields a Z of the correct weight.

Delft University of Technology

The maps g and h

The factoring maps are easily defined:

for $x \in X$ consider $\{Z \in \mathcal{Z} : x \in Z\}$; this is a filter on $\mathcal{Z}$ that is contained in a unique ultrafilter. That ultrafilter will be $g(x)$.

This is part of a general kind of duality between compact Hausdorff spaces and (normal, distributive) lattices: embeddings are dual to surjections and vice versa.